What are the Real numbers and how do we teach them in high school?

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What was the original question?

- Learn about the Real Numbers
    - The axioms
    - Applications
    - What makes the real numbers so special?
- Create a lesson plan
- Gain an understanding of what teaching is like
- Gain experience in the classroom

History

- 2000 BC - Natural Numbers ($\mathbf{N} = \{1, 2, 3....\}$), Babylonia
- 500 BC - Irrational Numbers (cannot be expressed as a fraction), Pythagoras
- 250 AD - Rational Numbers (fractions) fairly well
- 650 AD - Zero, India
- 700 AD - Negative Numbers, India
- 17th Century - Begin to use negative numbers

Familiarity with Real Numbers

Solutions to equations	
Equation	Solution over the reals
$x^2 + 4 = 0$	None
$x^2 - 5 = 0$	Two : $x = \pm\sqrt{5}$
$\sin x = 1$	Infinitely many : $x = \frac{\pi}{2} + 2n\pi \;\; n \in \mathbb{R}$
$(x^2 + 1)e^x = 0$	None:$e^x > 0$ for all x and $x^2 + 1 > 0$ for all x

Definitions

- Real numbers
- Field Axioms
  - Commutativity
  - Associativity
  - Existence of additive/multiplicate identities
  - Existence of additive/multiplicate inverses
  - Distributive law
- Order Axioms
  - Trichotomy
  - Transitivity
  - Addition/Multiplication in inequalities
- Completeness Axiom
  - Non-empty set bounded above
  - Supremum exists

Dedekind Cuts

- Richard Dedekind - 1858
- Dedekind left set
    - non-empty subset A of $\mathbb{Q}^+$
    - $A \neq \mathbb{Q}^+$
    - A is downward closed : If $a, b \epsilon \mathbb{Q}^+$, $b < a$, and $a \epsilon A$, then $b \epsilon A$
    - A has no largest element
- Dedekind positive real number, denoted $\mathbb{DR}^+$, is a Dedekind left set.
- $1^{\mathcal{R}} = \{x \in \mathbb{Q}^+ | x < 1\}$

Lemma 1

- $R, S \in \mathbb{DR}^+$
- $\mathbb{DR}^+$ is closed under both addition and multiplication
  - $R + S = \{r + s | r \in R, s \in S\}$
  - $R \cdot S = \{r \cdot s | r \in R, s \in S\}$
- Properties of rational numbers

Lemma 2

- Commutative property
  - $R + S = S + R$ for all R, S in $\mathbb{DR}^+$
  - Properties of rational numbers
- Associative Property
  - $(R + S) + T = R + (S + T)$ for all R, S, T in $\mathbb{DR}^+$
  - To prove, show if $a \subset b$ and $b \subset a$, then $a = b$
- Distributive property
  - $R \cdot (S + T) = (R \cdot S) + (R \cdot T)$ for all R, S, T in $\mathbb{DR}^+$
  - Similar to associative property

Identity Law

- $1^{\mathcal{R}} = \{x \in \mathbb{Q}^{+} | x < 1\}$
- $1^{\mathcal{R}}$ is a Dedekind positive real
- Downward closed

Other Definitions

- Order
  - $R < S$ if there exists a positive real number T such that $R + T = S$
- Subtraction
  - $R - S = \{r - s | r \in R \text{ and } s \notin S\}$ for all R and S such that this set in non-empty

Decimal Expansion

- $a = a_0.a_1a_2a_3\cdots$ where a_0 is any integer and $a_i, i = 1, 2, 3, \cdots$ is an integer between 0 and 9
- Regard $a_0.a_1a_2a_39999\cdots$ to be the same as $a_0.a_1a_2(a_3+1)0000\cdots$
- Minimal information

Cauchy Representation

- Only find definition of a cauchy sequence
  - for each positive ϵ there is an integer N such that:for all $n, m > N$, $|a_n - a_m| < \epsilon$
  - elements of the sequence are getting closer together
- Minimal information

State Standards

- Domain 1. Algebra 1.1 Algebraic Structures
  - a. " Know why the real (complex) numbers are a field and that particular rings are not fields (e.g. integers, polynomial rings, matrix rings)"
  - b. " Apply basic properties of real (complex) numbers in constructing mathematical arguments (e.g. if $a < b$ and $c < 0$ then $a \cdot c > b \cdot c$)"
  - c. " Know that the rational numbers and real numbers can be ordered"

Lesson

- Familiar number systems
    - Natural numbers
    - Integers
    - Rational numbers
    - Irrational numbers
- Introduce Real Numbers
- Properties (axioms)
- Worksheet

Worksheet

- Choose the equation that describes the given property.
- State which property of the real numbers is being used.
- Perform the given operations. Justify your answer using properties of the real numbers on the given handout.
- True or false in regards to order.
- Problems with false statements
  - $a * (b + c) = (a * b) + c$
- Problems with order
  - $\{7 < 11 \text{ and } 11 < 15\}$ then $15 < 7$

Evaluation

- Before today, how familiar were you with the topic?
- Did you learn any new concepts?
- What aspects of the lesson did you like?
- What aspects of the lesson could be improved?
- Give more examples
- Relax
- Board writing

Conclusion

- Enjoy teaching
- Trust myself
- More respect for teachers
- Grading is better than homework

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