

Permutation Statistics on Generalized Derangements

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- ▶ The order of a set G is the number of elements in the set, denoted as $|G|$
- ▶ $|S_n| = n!$

Cycle Notation

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in cycle notation.

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To determine the cycle notation for this permutation we put the permutation into 2-line notation.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 4 & 2 & 1 \end{pmatrix}$$

Cycle Notation

This shows us that the 1 is mapped to the 3, the 3 is mapped to the 6, and the 6 is mapped to the 1. This forms the cycle (1, 3, 6).

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$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 4 & 2 & 1 \end{pmatrix}$$

We start the next cycle with the 2, which is mapped to the 5. The 5 is mapped to the 2, giving us the second cycle (2, 5).

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$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 4 & 2 & 1 \end{pmatrix}$$

The 4 is mapped to itself giving (4). This is called a *fixed point*.

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Putting the cycles together gives $\sigma_2 = (1, 3, 6)(2, 5)(4)$

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$$\sigma_1 = 1\ 6\ 8\ 2\ 4\ 7\ 5\ 9\ 3 = (1)(2, 6, 7, 5, 4)(3, 8, 9)$$

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- ▶ D_n is the set of derangements in S_n

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is not a desarrangement because the leftmost increase is in the first position.

$$\sigma_2 = 7\ 5\ 4\ 2\ 3\ 9\ 8\ 6\ 1$$

is a desarrangement because the leftmost increase is in the fourth position.

Desarrangements

When a permutation has no increases, the leftmost increase is said to occur in the last position.

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$$\sigma_3 = 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1$$

is a not a desarrangement because the “leftmost increase” is in the ninth position.

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$$\sigma_4 = 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1$$

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- ▶ K_n is the set of desarrangements in S_n

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- ▶ $!n$ is the subfactorial of n

- ▶ $\lim_{n \rightarrow \infty} \frac{!n}{n!} = \frac{1}{e}$

Derangement to Desarrangement Mapping

To map a derangement to a desarrangement, the derangement is first put into cycle notation.

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The elements within each cycle are then rotated so that the **smallest element** is in the second position.

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$$(9, 3, 6)(5, 2, 4, 7)(8, 1)$$

Reading the numbers from left to right gives the desarrangement

$$9\ 3\ 6\ 5\ 2\ 4\ 7\ 8\ 1.$$

Derangement to Unknown Mapping

To map a derangement to an unknown, the derangement is again put into cycle notation.

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The elements within each cycle are then rotated so that the **second-smallest element** is in the last position.

$$(1, 8)(7, 5, 2, 4)(9, 3, 6)$$

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$$(1, 8)(7, 5, 2, 4)(9, 3, 6)$$

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$$(7, 5, 2, 4)(9, 3, 6)(1, 8)$$

Reading the numbers from right to left gives the unknown

$$8\ 1\ 6\ 3\ 9\ 4\ 2\ 5\ 7.$$

Inversion

- ▶ gives the total number of pairs for which the element on the left is greater than the element on the right

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Starting with the element in the first position we count the number of elements that are both to the right of and less than the element. Looking at 8 gives us 7 pairs.

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Starting with the element in the first position we count the number of elements that are both to the right of and less than the element. Looking at 8 gives us 7 pairs.

$$8 > 4$$

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$$8 > 4 \quad 8 > 6 \quad 8 > 7 \quad 8 > 2$$

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Starting with the element in the first position we count the number of elements that are both to the right of and less than the element. Looking at 8 gives us 7 pairs.

$$8 > 4 \quad 8 > 6 \quad 8 > 7 \quad 8 > 2 \quad 8 \not> 9$$

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$$8 > 4 \quad 8 > 6 \quad 8 > 7 \quad 8 > 2 \quad 8 \not> 9 \quad 8 > 5$$

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$$8 > 4 \quad 8 > 6 \quad 8 > 7 \quad 8 > 2 \quad 8 \not> 9 \quad 8 > 5 \quad 8 > 1 \quad 8 > 3$$

Looking at the next element, 4, gives an additional 3 pairs.

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$$4 \not> 6$$

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Looking at the next element, 4, gives an additional 3 pairs.

$$4 \not> 6 \quad 4 \not> 7$$

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$$4 \not> 6 \quad 4 \not> 7 \quad 4 > 2$$

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$$8 > 4 \quad 8 > 6 \quad 8 > 7 \quad 8 > 2 \quad 8 \not> 9 \quad 8 > 5 \quad 8 > 1 \quad 8 > 3$$

Looking at the next element, 4, gives an additional 3 pairs.

$$4 \not> 6 \quad 4 \not> 7 \quad 4 > 2 \quad 4 \not> 9 \quad 4 \not> 5$$

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$$4 \not> 6 \quad 4 \not> 7 \quad 4 > 2 \quad 4 \not> 9 \quad 4 \not> 5 \quad 4 > 1 \quad 4 > 3$$

Continuing this same process for each element except the last and summing the values gives

$$7 + 3 + 4 + 4 + 1 + 3 + 2 + 0 = 24$$

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- ▶ gives the total number of pairs for which the element on the left is greater than the element on the right

$$\sigma = 8\ 4\ 6\ 7\ 2\ 9\ 5\ 1\ 3$$

Starting with the element in the first position we count the number of elements that are both to the right of and less than the element. Looking at 8 gives us 7 pairs.

$$8 > 4 \quad 8 > 6 \quad 8 > 7 \quad 8 > 2 \quad 8 \not> 9 \quad 8 > 5 \quad 8 > 1 \quad 8 > 3$$

Looking at the next element, 4, gives an additional 3 pairs.

$$4 \not> 6 \quad 4 \not> 7 \quad 4 > 2 \quad 4 \not> 9 \quad 4 \not> 5 \quad 4 > 1 \quad 4 > 3$$

Continuing this same process for each element except the last and summing the values gives

$$7 + 3 + 4 + 4 + 1 + 3 + 2 + 0 = 24$$

So $\text{inv } \sigma = 24$.

Descent

- ▶ gives the total number of consecutive pairs for which the element on the left is greater than the element on the right

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$$\sigma = 8\ 4\ 6\ 7\ 2\ 9\ 5\ 1\ 3$$

Starting with the element in the first position we count the number of elements that are greater than the element immediately following.

Looking in the first position gives us $8 > 4$ which gives us 1.

Descent

- ▶ gives the total number of consecutive pairs for which the element on the left is greater than the element on the right

$$\sigma = 8\ 4\ 6\ 7\ 2\ 9\ 5\ 1\ 3$$

Starting with the element in the first position we count the number of elements that are greater than the element immediately following.

Looking in the first position gives us $8 > 4$ which gives us 1. Continuing through the remaining elements gives us 3 more.

$$4 \not> 6$$

Descent

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$$4 \not> 6 \quad 6 \not> 7$$

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$$4 \not> 6 \quad 6 \not> 7 \quad 7 > 2$$

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$$4 \not> 6 \quad 6 \not> 7 \quad 7 > 2 \quad 2 \not> 9$$

Descent

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This gives a total of

$$1 + 1 + 1 + 1 = 4$$

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This gives a total of

$$1 + 1 + 1 + 1 = 4$$

So $\text{des } \sigma = 4$.

Major Index

- ▶ gives the sum of the positions of the first element of each descent

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Starting with the element in the first position we find each element that is greater than the element immediately following and sum the positions of these elements.

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Continuing through the remaining elements gives us 3 more descents, at positions 4, 6, and 7.

$$4 \not> 6$$

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$$4 \not> 6 \quad 6 \not> 7 \quad 7 > 2 \quad 2 \not> 9 \quad 9 > 5 \quad 5 > 1 \quad 1 \not> 3$$

Summing the positions gives

$$1 + 4 + 6 + 7 = 18$$

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Summing the positions gives

$$1 + 4 + 6 + 7 = 18$$

So $\text{maj } \sigma = 18$

Q-Analog Equations

- ▶ If there are k permutations in S_n with a major index of j , then there are also k permutations in S_n with an inversion of j .

$$\sum_{\sigma \in S_n} q^{\text{maj } \sigma} = \sum_{\sigma \in S_n} q^{\text{inv } \sigma}$$

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$\sigma_S \in S_3$	$\text{inv } \sigma_S$	$\text{maj } \sigma_S$
1 2 3		
1 3 2		
2 1 3		
2 3 1		
3 1 2		
3 2 1		

$$\sum_{\sigma \in S_n} q^{\text{maj } \sigma} = \sum_{\sigma \in S_n} q^{\text{inv } \sigma} =$$

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$\sigma_S \in S_3$	$\text{inv } \sigma_S$	$\text{maj } \sigma_S$
1 2 3	0	0
1 3 2		
2 1 3		
2 3 1		
3 1 2		
3 2 1		

$$\sum_{\sigma \in S_n} q^{\text{maj } \sigma} = \sum_{\sigma \in S_n} q^{\text{inv } \sigma} = q^0 + \dots$$

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1 2 3	0	0
1 3 2	1	
2 1 3	1	1
2 3 1		
3 1 2		1
3 2 1		

$$\sum_{\sigma \in S_n} q^{\text{maj } \sigma} = \sum_{\sigma \in S_n} q^{\text{inv } \sigma} = q^0 + 2q^1 + \dots$$

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$\sigma_S \in S_3$	$\text{inv} \sigma_S$	$\text{maj} \sigma_S$
1 2 3	0	0
1 3 2	1	2
2 1 3	1	1
2 3 1	2	2
3 1 2	2	1
3 2 1		

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1 2 3	0	0
1 3 2	1	2
2 1 3	1	1
2 3 1	2	2
3 1 2	2	1
3 2 1	3	3

$$\sum_{\sigma \in S_n} q^{\text{maj } \sigma} = \sum_{\sigma \in S_n} q^{\text{inv } \sigma} = q^0 + 2q^1 + 2q^2 + q^3$$

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- ▶ If there are k permutations in D_n with a major index of j , then there are k permutations in K_n and U_n with an inversion of j .

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$\sigma_D \in D_4$	$\text{maj} \sigma_D$	$\sigma_K \in K_4$	$\text{inv} \sigma_K$	$\sigma_U \in U_4$	$\text{inv} \sigma_U$
2143		4321		4321	
2341		4123		2143	
2413		3124		2134	
3142		2134		2431	
3412		4231		4231	
3421		4132		2314	
4123		2143		2341	
4312		3142		2413	
4321		3241		4132	

$$\sum_{\sigma \in D_n} q^{\text{maj} \sigma} = \sum_{\sigma \in K_n} q^{\text{inv} \sigma} = \sum_{\sigma \in U_n} q^{\text{inv} \sigma} =$$

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2143		4321		4321	
2341		4123		2143	
2413		3124		2134	1
3142		2134	1	2431	
3412		4231		4231	
3421		4132		2314	
4123	1	2143		2341	
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4321		3241		4132	

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2143		4321		4321	
2341		4123		2143	2
2413	2	3124	2	2134	1
3142		2134	1	2431	
3412	2	4231		4231	
3421		4132		2314	2
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2143		4321		4321	
2341	3	4123	3	2143	2
2413	2	3124	2	2134	1
3142		2134	1	2431	
3412	2	4231		4231	
3421		4132		2314	2
4123	1	2143	2	2341	3
4312	3	3142	3	2413	3
4321		3241		4132	

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$\sigma_D \in D_4$	$\text{maj} \sigma_D$	$\sigma_K \in K_4$	$\text{inv} \sigma_K$	$\sigma_U \in U_4$	$\text{inv} \sigma_U$
2143	4	4321		4321	
2341	3	4123	3	2143	2
2413	2	3124	2	2134	1
3142	4	2134	1	2431	4
3412	2	4231		4231	
3421		4132	4	2314	2
4123	1	2143	2	2341	3
4312	3	3142	3	2413	3
4321		3241	4	4132	4

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2143	4	4321	6	4321	6
2341	3	4123	3	2143	2
2413	2	3124	2	2134	1
3142	4	2134	1	2431	4
3412	2	4231	5	4231	5
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K_n to U_n Mapping

$$\sigma_K = 6\ 2\ 3\ 4\ 1\ 5$$

K_n to U_n Mapping

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Unlike all the other mappings, the mapping from a desarrangement to an unknown does not start in cycle notation. Instead the permutation is divided into pieces such that the **smallest element** in each piece is in the 2nd position.

$$\sigma_K = [6\ 2\ 3][4\ 1\ 5]$$

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The **second-smallest** element in each piece is then identified.

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The **second-smallest** element in each piece is then identified.

$$\sigma_K = [6\ 2\ 3][4\ 1\ 5]$$

Now the unknown begins being created. Keeping the same piece divisions as in the desarrangement, the second-smallest elements are placed in the first position of each piece in decreasing order.

$$\sigma_K = [6\ 2\ 3][4\ 1\ 5]$$

$$\sigma_U = [4\ -][3\ -]$$

K_n to U_n Mapping

We then look at the non-set elements in each permutation.

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the pieces are determined by the **smallest elements** the **1** and the **2**, so these are set elements.

In the unknown

$$\sigma_U = [4 \ -][3 \ -]$$

the **second smallest** elements from each desarrangement piece are set, so **3** and **4** are the set elements.

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$$\sigma = 4\ 3\ 8\ 5\ 1\ 9\ 6\ 2\ 7 = (1, 4, 5)(2, 3, 8)(6, 9, 7)$$

To map the 2-derangement to a 2-desarrangement the elements within each cycle are rotated so that the **smallest element** is in the 3rd $(2 + 1)$ position.

$$(4, 5, 1)(3, 8, 2)(9, 7, 6)$$

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Reading the numbers from left to right gives the 2-desarrangement

$$9\ 7\ 6\ 3\ 8\ 2\ 4\ 5\ 1.$$

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 - ▶ The 2 elements preceding the element in the 3rd position, 6, are greater than 6

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- ▶ The final two conditions are extensions of the fourth condition that do not apply in this case. They guarantee that the permutation could have come from an m -derangement.

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- ▶ Look at m -derangements and m -desarrangements of signed permutations

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