Voting Theory

Jessica Newville

California Lutheran University

5.16.08

The Big Question

I am interested in finding out whether bloc versus non-bloc voting of CLU faculty will affect the distribution of power in yes-no voting systems within the faculty community as measured by the Shapley-Shubik power index.

Shapley-Shubik Index of Power

- The amount the vote is worth is called the *weight* of the vote.
- A *winning coalition* is a collection of voters where passage of an issue is guaranteed because the weight of their votes exceeds the number of votes need for a win.
- **Definition:** The *pivotal person* for an ordering is the one whose joining converts a coalition from a non-winning coalition to a winning coalition.
- **Definition:** Suppose p is a voter in a yes-no voting system and let X be the set of all voters. Then the *Shapley-Shubik index of power* for p, denoted by $SSI(p)$, is the number given by:

$$SSI(p) = \frac{\text{the number of orderings of X for which p is pivotal}}{\text{the total number of possible orderings of the set X}}$$

- **Example:** Suppose we have a three person weighted voting system in which x has 50 votes, y has 49 votes and z has 1 vote. Assume 51 votes are needed for a win,[51: 50, 49, 1]. The 6 possible orderings with the pivotal player in bold are:

x **y** z

x **z** y

y **x** z

y z **x**

z **x** y

z y **x**

- Shapley-Shubik index of power of x, y and z:
$SSI(x) = \frac{4}{6}$
$SSI(y) = \frac{1}{6}$
$SSI(z) = \frac{1}{6}$

Bloc Voting

- Bloc voting occurs when the members of a party do not vote individually as they like. Rather, the party must collectively decide on one single way to vote.

Example

- Consider the previous example where [51: 50, 49, 1].
- If x and z vote in a bloc, then we would have two voters xz and y. However, xz has enough votes to win, making xz always pivotal.
- Therefore the $SSI(xz) = 1$; and as you can see xz now has a large influence in this vote.

Bloc Voting and CLU

- CLU faculty can be divided into three groups, where each group has the potential to bloc vote.
- The CLU groups:

  Professors: 49
  Associate Professors: 24
  Assistant Professors/Instructors: 47

- Excluding Senior Lectors who have faculty voice but no vote.
- Excluding anyone who is on sabbatical for the whole year.

Shapley-Shubik and CLU: When One Group Bloc Votes

Theorem

Suppose we have M players, V voters, and that a single bloc **B** *of size B forms. The resulting weighted voting body is:*

[Q:B, 1, 1, . . . ,1], where there are M-B voters of weight one.

Then whenever $B < Q < M - B + 1$*, the Shapley-Shubik index of the bloc* **B***, denoted SSI(***B***), is given by:*

$$SSI(\mathbf{B}) = \frac{B}{M-B+1} = \frac{B}{V}$$

Shapley-Shubik and CLU: When One Group Bloc Votes

Proof.

- Initially there are M players, so $V = M - B + 1$ voters. Thus there are $V!$ distinct orderings for a coalition.
- The earliest B could be pivotal is when it is the $(Q - B) + 1$ player to join the coalition, since the number of votes would total the number of voters with weight 1 because $(Q - B) + B = Q$.
- The latest **B** could join the coalition and still be pivotal is in the Qth spot.
- Thus **B** is pivotal at the spots:

$$Q - 1, Q - 2, ..., Q - B.$$

- As we can see there are exactly B spots in the list above. Therefore, the Shapley-Shubik index of a bloc of size B is given by:

$$SSI(B) = \frac{\text{number of orderings in which B is pivotal}}{\text{total number of distinct orderings}} = \frac{(B)(V-1)!}{V!} = \frac{B}{V}.$$

□

Shapley-Shubik and CLU: When One Group Bloc Votes

Theorem

Suppose we have M players and that a single bloc of size B forms. Then the SSI of an unbloced group **A** *of size A is given by:*

$$SSI(\mathbf{A}) = \frac{A(V-B)}{V(V-1)}.$$

Shapley-Shubik and CLU: When Two Groups Bloc Vote

Proof.

- We want a member from **A** to join the coalition in the pivotal position. This means there are $\binom{A}{1}$ ways this will happen.
- If **A** is pivotal then **B** can not be pivotal. We know that **B** can join the coalition and be pivotal in B ways from Theorem 1, thus $V - B$ ways that **B** will not be pivotal.
- Finally, place the remaining voters in $(V-2)!$ ways and divide out by the total number of voters, $V!$ this gives:

$$\begin{aligned} SSI(\mathbf{A}) &= \frac{(V-B)(\binom{A}{1})(V-2)!}{V!} \\ &= \frac{(V-B)(A)}{(V)(V-1)}. \end{aligned}$$

□

Shapley-Shubik and CLU: When Two Groups Bloc Vote

Theorem

Suppose we have M players and V voters. Suppose that two blocs of size A and B form. Then the power of a bloced group of size B is:

$$SSI(\mathbf{B}) = \frac{\sum_{s=max[2,(Q-A-B+2)]}^{Q-A+1} (s-1) + \sum_{s=Q-B+1}^{min[Q,(V-1)]} (V-s)}{(V-1)(V)}$$

Shapley-Shubik and CLU: When Two Groups Bloc Vote

Theorem

When bloced groups **A** *and* **B** *form of size A and B, the power of the remaining unbloced group* **C** *of size C is:*

$$\begin{aligned} SSI(\mathbf{C}) &= \frac{(C)(Q-A-B)(Q-A-B-1)}{(V)(V-1)(V-2)} \\ &+ \frac{(C)(Q-A)(V-Q+A-1)}{(V)(V-1)(V-2)} \\ &+ \frac{(C)(Q-B)(V-Q+B-1)}{(V)(V-1)(V-2)} \\ &+ \frac{(C)(V-Q)(V-Q-2)}{(V)(V-1)(V-2)}. \end{aligned}$$

Movement Diagrams

- Once we have the values of the power index for the various groups we can represent the power for the voting system as an ordered triple.
- Professors (P), Associates (S) and Asistants/Instructors (A)
- When no one blocs:

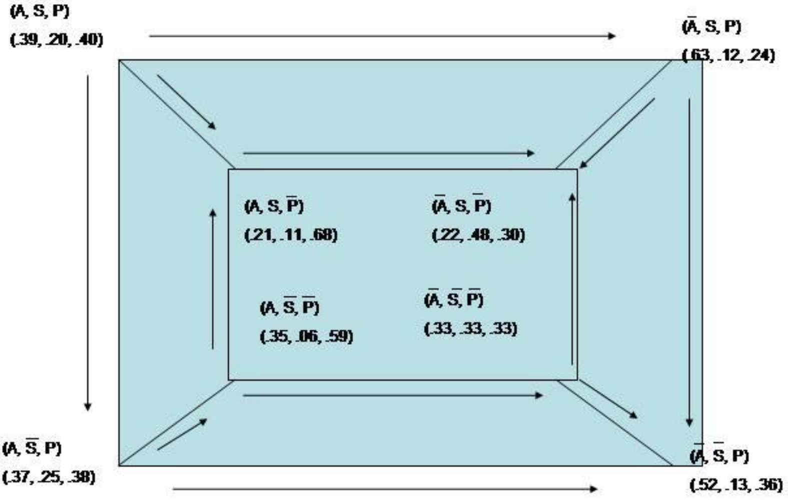
$$\begin{aligned} \text{SSI(A)} = .39,\ \text{SSI(S)} = .20 \text{ and } \text{SSI(P)} = .40 \\ = (.39,\ .20,\ .40) \end{aligned}$$

- When A blocs, denoted $\bar{A}$:

$$\begin{aligned} \text{SSI}(\bar{A}) = .63,\ \text{SSI(S)} = .12 \text{ and } \text{SSI(P)} = .24 \\ = (.63,\ .12,\ .24) \end{aligned}$$

- The eight different voting systems derived from the orderings of the three groups bloc voting or not bloc voting can be arranged on the corners of a cube.
- Each edge represents a change in the status of just one group.
- We associate with each edge an arrow indicating which state the affected group would prefer and if all other groups did not change their state.

Movement Diagram: (Assistants, Associates, Professors)



Further Research

- I need one more theorem for when three players bloc vote.
- I need one more theorem for when four players bloc vote.
- Construct a movement diagram for a 4 player voting system.

- Taylor, Alan D. *Mathematics and Politics: Strategy, Voting, Power and Proof*. New York: Springer-Verlag, 1995. 63-80.

  I used this book for my definitions and examples of Shapley-Shubik and Banzhaf power indices.

- Ross, Sheldon. *First Course in Probability* . 7th ed. Upper Saddle River: Prentice Hall, 2006. 3-7.

  This was my text book for my probability class at Channel Islands and I used it to brush up on combinations and Permutations.

- Straffin Jr., Philip D. "The Power of Voting Blocs: an Example." Mathematics Magazine 50 (1977): 22-24.
  I will use this to study block voting.

- Dr. Fogel, December 2007 - March 2008

  Dr. Fogel directed me in the right path for my objectives and helped me narrow down many ideas. She also reviewed and critique my objectives and presentation.