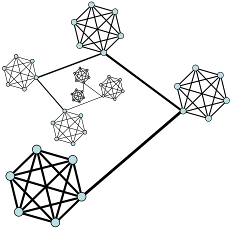
Minimal k-rankings of graphs



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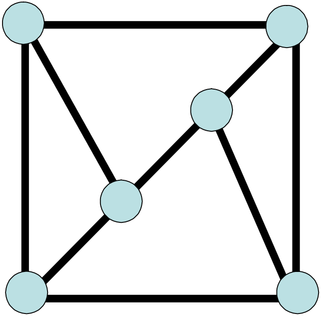
A quick introduction to Graph Theory

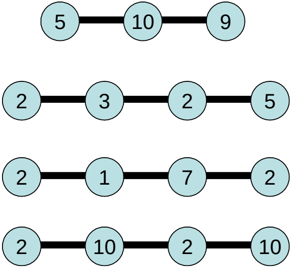
A graph, G, consists of a set of vertices, $V(G)$, and a set of edges where an edge connects two vertices.

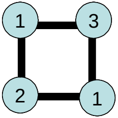


Ranking condition:

Definition

Given a graph G, a function $f : V(G) \longrightarrow \{1, 2, 3, ..., k\}$ is a k-ranking of G if $f(u) = f(v)$ implies every uv path contains a vertex w such that $f(w) > f(u)$.





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A k-ranking is minimal if the reduction of any label greater than 1 violates the described ranking condition.

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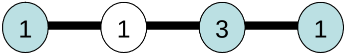

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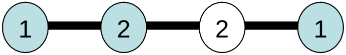

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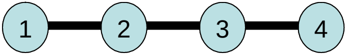

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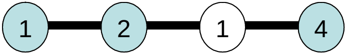

Minimal

Definition

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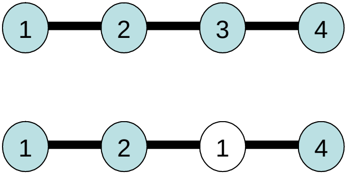
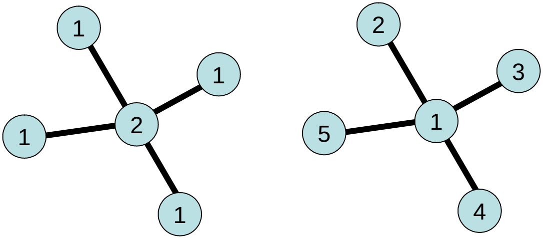
Hence, this labeling of a path on 4 vertices is not minimal.



Rank and a-rank number

Definitions

The rank number of a graph, denoted $\chi_r(G)$, is the smallest k such that G has a minimal ranking. The $a - rank$ number of a graph, denoted $\psi_r(G)$, is the largest k such that G has a minimal ranking.



Graph product

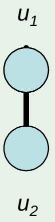
Definition

In general, a graph product of two graphs G_1 and G_2 is a new graph whose vertex set is $V(G_1) \times V(G_2)$ and where the adjacency of the new vertices is dependent on a certain condition. The most common graph product is the graph Cartesian Product.

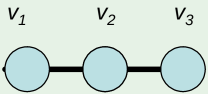
The graph Cartesian product

Example

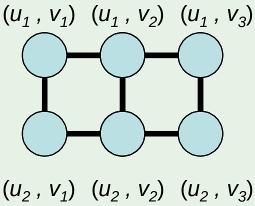
The graph Cartesian product of a path on 2 vertices and a path on 3 vertices is the following:



P_2



P_3



$P_2 \times P_3$

Cool facts about the Cartesian product.

1. The Cartesian product has been widely investigated, has numerous interesting algebraic properties, and is considered to be the easiest graph product.

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3. The Cartesian product is both commutative and associative.
4. A graph is considered to be prime with respect to the Cartesian product if the identity $G = G_1 \square G_2$ implies that G_1 or G_2 is the one-vertex graph K_1.

Super cool fact: Prime decomposition of graphs

Theorem

Every graph G has a prime factor decomposition with respect to the Cartesian product. The number of prime factors is at most $log_2|G|$.

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Every graph G has a prime factor decomposition with respect to the Cartesian product. The number of prime factors is at most $\log_2 |G|$.

Theorem (Sabidussi-Vizing)

Every connected graph G has a unique prime factor decomposition with respect to the Cartesian product.

Lemma (Ortiz)

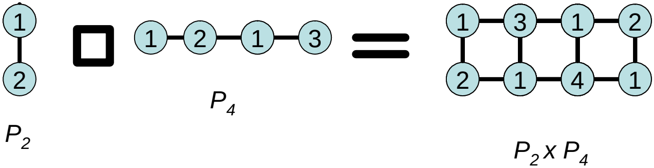
If $G = G_1 \Box G_2 \Box G_3 \ldots \Box G_k$, *where* G_i *are non-trivial graphs, then* $\chi_r(G) \geq \max\{\chi_r(G_i)\}$, *where* $1 \leq i \leq k$.

Proof.

If H is a subgraph of a graph G, then $\chi_r(G) \geq \chi_r(H)$. Since G_i is a subgraph of G, $\chi_r(G) \geq \chi_r(G_i)$, for all $1 \leq i \leq k$. Hence $\chi_r(G) \geq \max\{\chi_r(G_i)\}$. □

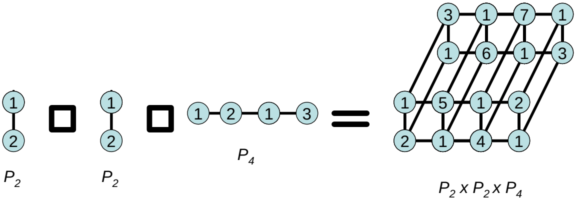
Lemma (Ortiz)

If G is the graph Cartesian product of G_1 and G_2, where G_1 and G_2 are not K_1, then $\chi_r(G) \geq \max\{\chi_r(G_1), \chi_r(G_2)\} + 1$

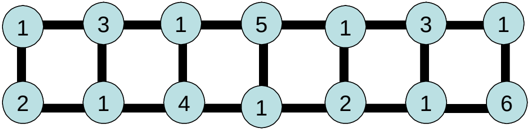


Lemma (Ortiz)

If $G = G_1 \square G_2 \square G_3 \ldots \square G_k$, *where* G_i *are non-trivial graphs for* $1 \leq i \leq k$, *then* $\chi_r(G) \geq \max\{\chi_r(G_i)\} + (k-1)$.



Rank number of a ladder



Let L_n be a ladder graph and P_n is a path on n vertices.

$$\chi_r(L_n) = \psi_r(P_n) + 1 =$$
$$\lfloor \log_2(n+1) \rfloor + \left\lfloor \log_2(n+1-(2^{\lfloor \log_2 n \rfloor - 1})) \right\rfloor + 1$$

(Narayan, Novotny, Ortiz, 2007)

Nicer Formula

Old version: $\chi_r(L_n) = \psi_r(P_n) + 1 =$
$\lfloor \log_2(n+1) \rfloor + \left\lfloor \log_2(n+1-(2^{\lfloor \log_2 n \rfloor - 1})) \right\rfloor + 1$

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Theorem (Ortiz)

$\chi_r(L_n) = 2 \cdot \lfloor \log_2 n \rfloor + r$, *where*

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Theorem (Ortiz)

$\chi_r(L_n) = 2 \cdot \lfloor \log_2 n \rfloor + r$, *where*
$r = 2$, *if* $n = 2^k - 1$,
$r = 0$, *if* $n = 2^k + x$, *where* x *is an integer in* $[0, 2^{k-1} - 2]$, *and*

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Old version: $\chi_r(L_n) = \psi_r(P_n) + 1 = \lfloor \log_2(n+1) \rfloor + \left\lfloor \log_2(n+1-(2^{\lfloor \log_2 n \rfloor - 1})) \right\rfloor + 1$

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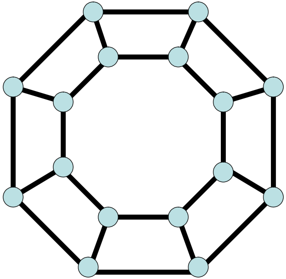
$r = 0$, *if* $n = 2^k + x$, *where* x *is an integer in* $[0, 2^{k-1} - 2]$, *and*
$r = 1$, *if* $n = 3 \cdot 2^k - 1 + y$, *where* y *is an integer in* $[0, 2^{k-1} - 1]$.

Lower bound for the *rank* number of Prism graphs

Theorem (Ortiz)

Let $Y_n = C_n \square P_2$ *be a prism graph such that* $n = 2^k + x$, *where* x *is an integer in* $[0, 2^{k-1} - 2]$, *or* $n = 3 \cdot 2^{k-1} + y$, *where* y *is an integer in* $[0, 2^{k-1} - 1]$ *and* $n \geq 4$. *Then* $\chi_r(L_n) + 1 \leq \chi_r(Y_n)$.



$Y_8 = C_8 \square P_2$

Upper bound for the *rank* number of Prism graphs

Theorem (Ortiz)

If $Y_n = C_n \Box P_2$ *is a prism graph, then* $\chi_r(Y_n) \leq \chi_r(L_{n-2}) + 2$*, for* $n \geq 4$.

Proof.

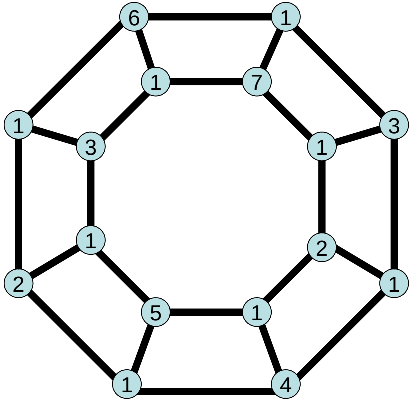
Since L_{n-2} is a subgraph of Y_n, we can label it with a $k-$ranking, where $\chi_r(L_{n-2}) = k$. We label the remaining 4 vertices with the labels 1, 1, $k+1$, $k+2$. Thus the vertices of Y_n can be labeled using $k+2$ labels. So $\chi_r(Y_n) \leq \chi_r(L_{n-2}) + 2$. □

The Rank number of Prism graphs

Corollary

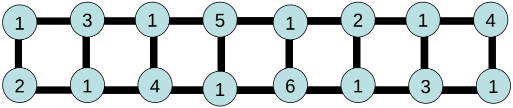
If $Y_n = C_n \square P_2$ *is a prism graph and* $n = 2^k$ *or* $n = 2^k - 2^{k-2}$*, then* $\chi_r(Y_n) = \chi_r(L_{n-2}) + 2$

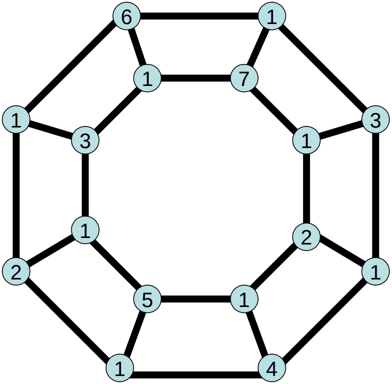
The *rank* number of Prism graphs



$Y_8 = C_8 \,\square\, P_2$

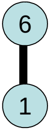
Since L_8 is a subgraph of Y_8, $\chi_r(Y_8) \geq \chi_r(L_8) = 6$.



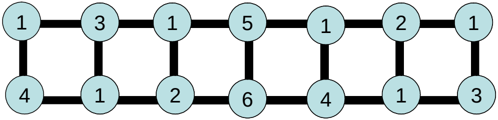


The above labeling of Y_8 gives an upper bound for $\chi_r(Y_8)$; namely $\chi_r(Y_8) \leq 7 = \chi_r(Y_6) + 2$. So $\chi_r(Y_8)$ is either 6 or 7. For the sake of contradiction, assume that $\chi_r(Y_8) = 6$.

Since the highest label in labeling of Y_8 is unique, we would have labeled $Y_n - (L_{n-1})$ as such:

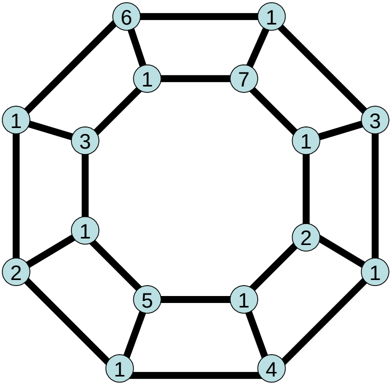


This leaves the labels 1-5 to label L_7.



This leads to a contradiction since $\chi_r(L_7) = 6$. Hence $7 = \chi_r(L_8) + 1 \leq \chi_r(Y_8) \leq \chi_r(Y_6) + 2 = 7$. Thus $\chi_r(Y_8) = 7$.

In General...



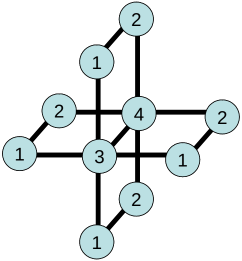
Corollary

If $Y_n = C_n \Box P_2$ *is a prism graph and* $n = 2^k$ *or* $n = 2^k - 2^{k-2}$, *then* $\chi_r(Y_n) = \chi_r(L_{n-2}) + 2$

The rank number of a book graph has been found!

Theorem (Ortiz)

Let $B_n = S_n \square P_2$ be a book graph with $n \geq 3$, where S_n is a star graph on n vertices. Then $\chi_r(B_n) = 4$.

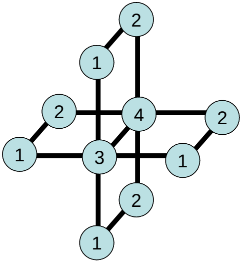


From a previous lemma
$\chi_r(B_n) \geq \max\{\chi_r(S_n), \chi_r(P_2)\} + 1 = \max\{2, 2\} + 1 = 3$. The above labeling of B_4 gives an upper bound for $\chi_r(B_4)$; namely $\chi_r(B_4) \leq 4$. So $\chi_r(B_4)$ is either 3 or 4.

The rank number of a book graph was found!

Theorem (Ortiz)

Let $B_n = S_n \square P_2$ is a book graph and $n \geq 3$, where S_n is a star graph on n vertices. Then $\chi_r(B_n) = 4$.

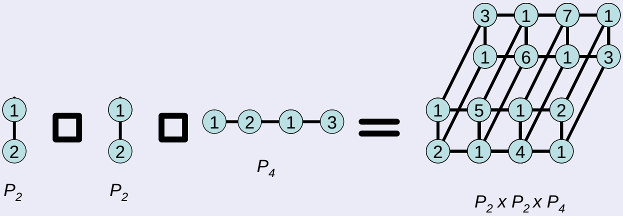


However, $\chi_r(H) \leq \chi_r(B_4)$ for any subgraph H of B_4. Since $P_3 \square P_2$ is a subgraph of B_n. It follows that $4 = \chi_r(P_3 \square P_2) \leq \chi_r(B_4) \leq 4$. Thus $\chi_r(B_4) = 4$

Problem for everyone!

Problem

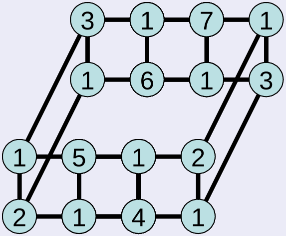
Why is $\chi_r(P_2 \square P_2 \square P_4) = 7$?



Answer

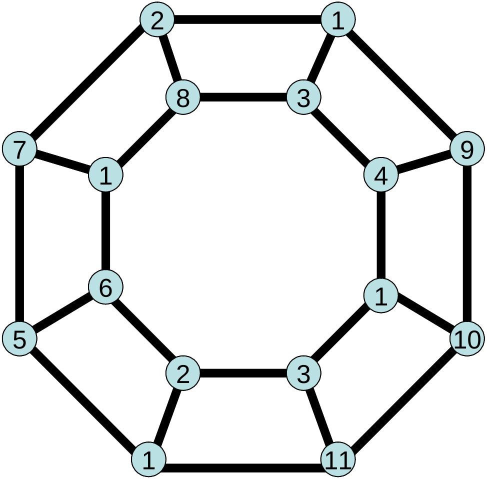
Solution

$\chi_r(P_2 \Box P_2 \Box P_4) \geq \chi_r(P_2 \Box P_8) = 7.$



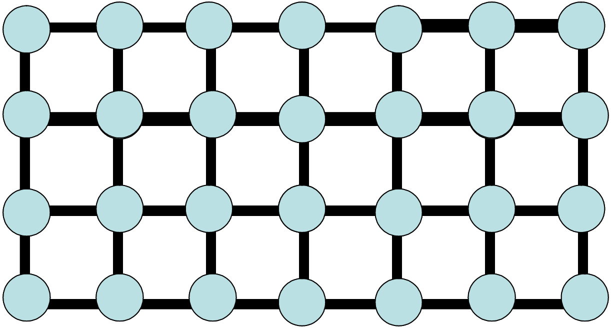
Future Work:

Find the a-rank number of a prism graph



Future Work:

Find the rank and a-rank number of a grid graph



References

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