Schwarzschild Geometry in association with Black Holes

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History

- "A *black hole* is an astronomical contradiction - a dark star, an invisible nothing, a prison of light."
- A black hole - an object predicted by general relativity with a gravitational field so powerful that even light cannot escape its pull.
- In 1784 an amateur astronomy, John Mitchell, brought the idea of black holes back into the mainstream.
  - He used Newton's theory of gravity to prove that for some dense dark stars, the escape velocity would be greater than the speed of light.
- The interest of black holes continued in 1915 with Einstein's theory of relativity, where he predicted certain singularities which he called "Schwarzschild singularities."

A Little Background

- The Schwarzschild Metric: an important metric in general relativity that describes the geometry of a black hole.
- Fundamental ideas: 1) gravity is geometry and 2) gravity occurs from the curvature of spacetime
- spacetime: four dimensional containing the three spatial coordinates (r, θ, ϕ) and one temporal coordinate (t)
- Einstein proposed the idea that "the Earth moves in its orbit around the sun, not because a force of gravity acts on it, but because it is following the straightest possible path in the slightly non-Euclidean geometry produced by the Sun." -Harle
  - he suggested that mass curves the geometry of spacetime nearby

Definitions

- *escape velocity* - minimum speed an object needs to escape the gravitational clutches of a planet or star
- *singularity* - a point at the very center of a black hole at which gravitational forces cause matter to have infinite density, no volume, and distorted space and time
- *metric* - a geometric function that describes the distances between pairs of points in a space - American Heritage Dictionary

Schwarzschild Geometry

- the geometry of empty space outside a spherically symmetric source of curvature (i.e. a spherical star)
- The metric associated with the Schwarzschild geometry is given by: $ds^2 = -(1 - \frac{2GM}{c^2 r})(cdt)^2 + (1 - \frac{2GM}{c^2 r})^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$
- This formula can be expressed in a simpler form by rescaling the variables c and G to equal 1 in the above equation, so that the final equation has the form: $ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$
  - corresponding coordinates: Schwarzschild coordinates
  - corresponding metric: Schwarzschild metric

Properties of the Schwarzschild Metric

- *ds* - invariant spacetime interval, an absolute measure of the distance between two events in space and time
- *Time Element* - the metric is independent of time
- *Line Element* - $ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2)$ where $t = r = constant$ therefore $dt = dr = 0$
- *Spherically Symmetric*
- *Mass Element* - the constant M in the Schwarzschild metric is identified with the total mass of the source
- *Schwarzschild Radius*

Spherically Symmetric

- From the geometry of a two-dimensional surface, the line element describes the geometry of a sphere of radius r.
- The line element gives the symmetries of a sphere with regards to changes in the angles θ and ϕ.
- The Schwarzschild coordinate r is related to the area A of the two-dimensional spheres with a fixed r and t by the formula: $r = (A/4\pi)^{1/2}$.

Schwarzschild Radius

- usually denoted by the expression: R_s
- given by the formula: $R_s = 2GM/c^2$
- This equation was derived from John Mitchell's equation for escape velocity: $v_{escape}^2 = 2GM/R_s$
- Also known as the event horizon - the outer edge of the black hole
- The surface of a star with the Schwarzschild Geometry, joins a different coordinate system inside the star, Eddington-Finkelstein coordinates.

Eddington-Finkelstein Coordinates

- By observing the equation
  $ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$
- we can see that the Schwarzschild metric has singularities at $r \leq 2M$, therefore a new coordinate systems must be introduced.
- This coordinate system is called the Eddington-Finkelstein coordinates (i.e. E-F coordinates) which does not have the singularity inside the black hole.
- These coordinates are used to describe the physics of a black hole outside, at and inside the Schwarzschild radius.

E-F Coordinates continued

- To convert the Schwarzschild coordinates into the E-F coordinates, the time coordinate t is modified in terms of a new coordinate v defined by: $t = v - r - 2M \log |\frac{r}{2M} - 1|$
- By taking the derivative, then squaring the derivative, and finally cancelling the like terms, we then insert this value into the dt^2 of the Schwarzschild metric in order to get our final equation:
- $ds^2 = -(1 - \frac{2M}{r})dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2)$
- The E-F coordinates only have the singularities at $r = 2M$ and $r = 0$
- These coordinates are functional in studying the *ingoing* and *ongoing* gravitational collapse of a black hole.
  - This function is better explained through Light Cones.

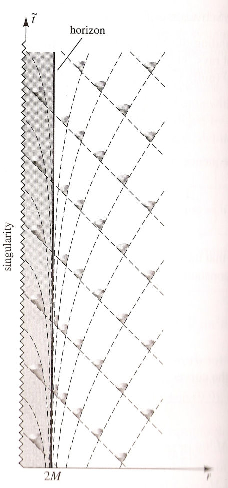
Light Cones of the Schwarzschild Geometry

- To get a better understanding of the Schwarzschild geometry, we need to analyze the behavior of radial light rays.
- Light rays move along the lines for which $\mathrm{d}\theta = \mathrm{d}\phi = 0$ and $ds^2 = 0$.
- By plugging these value into the E-F coordinates and simplifying, we get: $-(1 - \frac{2M}{r})dv + 2dr = 0$
- Solving for dv/dr, integrating with respect to r, and simplifying, we get: $v - 2(r + 2M \ln|\frac{r}{2M} - 1|) = const$

Light Cones continued

- RADIAL RAY 1 - $v = constant$
- RADIAL RAY 2 - $v - 2(r + 2M \ln|\frac{r}{2M} - 1|) = const$
- *Case 1*: $r > 2M$. Ray 1 moves inward to smaller and smaller values of r where as the ray 2 moves outward to larger and larger values of r. This situation is known as *outgoing* radial rays.
- *Case 2*: $r < 2M$. This is where both rays move inward and eventually hit the singularity $r = 0$. This situation is known as *ingoing* radial rays
- *Case 3*: $r = 2M$. This surface is called the event horizon. The surface divides the spacetime into two regions: 1) light can escape to infinity and 2) gravity is so strong that not even light can escape.

Radial Light Rays in E-F Coordinates



- *outgoing* $(r > 2M)$: rays that spread outwardly to infinity
- *ingoing* $(r < 2M)$: rays collapse inward to the singularity $r = 0$ as r decreases and v increases
- *horizon* $(r = 2M)$: the heavy vertical line that divides the region where light rays neither fall into the singularity nor escape to infinity

Why is this significant?

- The Schwarzschild geometry is used to help solve applications in gravitational lensing, X-ray binaries, and finding black holes in the center of our galaxy.
- However, the mathematics and physics used in solving these applications go well beyond my knowledge.
- I hope that in the future I will be able to apply my research and background of the Schwarzschild geometry to my graduate work in astronomy.

Bibliography

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