The Box Problem

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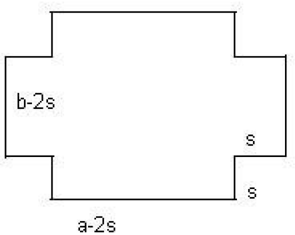
A box with an open top is to be constructed from a square piece of cardboard with dimensions, a x b, where a and b are rational. The student is asked to find the height of the box that gives the largest volume.

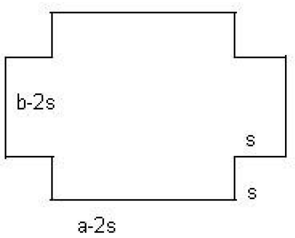


Student Solution: Maximize the Volume

$V(s) = (a - 2s)(b - 2s)s = 4s^3 - 2(a + b)s^2 + abs.$

$V'(s) = 12s^2 - 4(a + b)s + ab.$

We need $V'(s) = 0$.



Solution cont.

The solution is $s = \frac{4(a+b) \pm \sqrt{16(a+b)^2 - 48ab}}{24}$.

We want the solution s to be rational so the discriminant $16(a+b)^2 - 48ab$ needs to be a perfect square.

In other words, $(a+b)^2 - 3ab = a^2 - ab + b^2$ needs to be a perfect square.

Then the solution, s, will be rational if the dimensions of the box, a and b, satisfy the equation

$$a^2 - ab + b^2 = c^2,$$

where c is rational.

Notice this is the definition of an Eisenstein Triple, (a, b, c).

Hotchkiss' Theorem

Theorem

Let a and b be rational numbers. Then the solution to the Box Problem occurs at a rational value of s if and only if $a = r(1 - m^2)$ and $b = r(2m - m^2)$ where r and m are rational numbers with $0 < m < 1$.

Example: Let's choose r=2 and m=1/3.

$a = 2(1 - \frac{1}{9})$ and $b = 2(\frac{2}{3} - \frac{1}{9})$

The dimensions of the rectangle are $\frac{16}{9}$ x $\frac{10}{9}$.

Notice that Hotchkiss' Theorem produces rational dimensions as well as rational solutions for s.

Open Questions

Open Question 1

Is there a recursive formula that generates all possible values of a and b that are solutions to the Box Problem?

Open Question 2

Is $a = 3$ and $b = 8$ the smallest pair of distinct integers that give a rational solution to the Box Problem?

Open Question 3

Is $a = 5$ and $b = 8$ the smallest pair of distinct integers that give an integer solution to the Box Problem?

Open Question 1

Marissa's Theorem 1

Let (a_n, b_n, c_n) be an Eisenstein triple and a_n is odd. Then the recursion
$a_{n+1} = 2 + a_n$,
$b_{n+1} = 2 + a_n + b_n$,
$c_{n+1} = \sqrt{4 + 4a_n + a_n{}^2 + 2b_n + a_n \cdot b_n + b_n{}^2}$, generates Eisenstein triples.

By plugging a_{n+1} and b_{n+1} into $a^2 - ab + b^2 = c^2$,

$$a_{n+1}{}^2 - a_{n+1}b_{n+1} + b_{n+1}{}^2 =$$

$$(2 + a_n)^2 - (2 + a_n)(a_{n+1} + b_n) + (a_{n+1} + b_n)^2 =$$

$$4 + 4a_n + a_n{}^2 + 2b_n + a_n \cdot b_n + b_n{}^2.$$

This means $c_{n+1} = \sqrt{4 + 4a_n + a_n{}^2 + 2b_n + a_n \cdot b_n + b_n{}^2}$ and therefore $(a_{n+1}, b_{n+1}, c_{n+1})$ is an Eisenstein Triple.

Open Question 2

Open Question 2

Is $a = 3$ and $b = 8$ the smallest pair of distinct integers that give a rational solution to the Box Problem?

Marissa's Theorem 2

If a and b are distinct integers that give a rational solution to the Box Problem, then $(a > 3)$ or $(a = 3$ and $b \geq 8)$.

Sketch of Proof:

Step 1: Show $a \neq 1$ and $a \neq 2$ by finding a contradiction.

Step 2: Check the pairs (3,1), (3,2), (3,4), (3,5), (3,6), (3,7).

Open Question 3

Open Question 3

Is $a = 5$ and $b = 8$ the smallest pair of distinct integers that give an integer solution to the Box Problem?

We now want the solution s to be an integer, but Hotchkiss' Theorem gives us dimensions (a, b) for rational solutions. Hence the following theorem:

Hotchkiss' Theorem 2

Let a and b be rational numbers. Then the maximum volume of the box constructed in the Box Problem occurs at an integral value of s if and only if $a = r(1 - m^2)$, $b = r(2m - m^2)$, and $r(m - m^2)$ is an even integer, where r and m are rational numbers with $0 < m < 1$.

Marissa's Theorem 3

If a and b are distinct integers that give an integral solution to the Box Problem, then $(a > 5)$ or $(a = 5$ and $b \geq 8)$.

Proof: First we need to show that $a > 5$:

$a \neq 1$ and $a \neq 2$ have already been shown in Theorem 2.

Let us look at when $a = 3$.

Proof of Theorem 3

Show $a \neq 3$

When $a = 3$, suppose for the sake of contradiction that b is an integer other than 3.

Assume that $r(m - m^2)$ is an even integer, so that the solution is an integer.

From Hotchkiss' Theorem 2, let $3 = r(1 - m^2)$ where r and m are rational and $0 < m < 1$.

Thus,

$$r = \frac{3}{1 - m^2}.$$

Then by substitution, $r(m - m^2) = \frac{3m}{1+m}$.

Proof of Theorem 3

Show $a \neq 3$

Remember $r(m - m^2) = \frac{3m}{1+m}$ and $r(m - m^2)$ is even.

Define $m = \frac{u}{v}$ where $u, v \epsilon \mathbf{Z}^+$, relatively prime, with $u < v$.

Then,

$$r(m - m^2) = \frac{3\frac{u}{v}}{1 + \frac{u}{v}} = \frac{3u}{v + u} = 2n,$$

where n is a positive integer.

Solving for u, $u = \frac{2nv}{3-2n}$.

Since u, v, and n are all positive integers, $n = 1$.

Then $u = 2v$ so $m = \frac{u}{v} = 2$.

But, Hotchkiss' Theorem requires that $0 < m < 1$.

Therefore $a \neq 3$.

Marissa's Lemma

If $a = 4$ and the solution s is rational, there are no distinct integer solutions for b.

Sketch of Proof:

When $a = 4$, $\sqrt{16 - 4b + b^2}$ must be rational.

Assume it is rational and find a contradiction.

Therefore by Marissa's Lemma, $a \neq 4$.

Marissa's Theorem 3

If a and b are distinct integers that give an integral solution to the Box Problem, then $(a > 5)$ or $(a = 5$ and $b \geq 8)$.

We have shown that $a > 5$.

Now we must show that the pairs (a, b) are not possible dimensions for the box problem: (5,1), (5,2), (5,3), (5,4), (5,6), (5,7).

Checking pairs (a, b)

Let $a = r(1 - m^2)$ and $b = r(2m - m^2)$.

By solving the first equation for r, $r = \frac{a}{1-m^2}$

By substitution, $b = \frac{a(2m-m^2)}{(1-m^2)}$.

Thus,

$$(1 - m^2)b = a(2m - m^2) \quad (1)$$

where m is rational and $0 < m < 1$.

Let us check the pair (5,1).

By plugging into Equation (1), $1 - m^2 = 10m - 5m^2$.

Solving for m, we get $m = \frac{10 \pm \sqrt{84}}{8}$, which is not rational so (5,1) are not possible dimensions.

Checking pairs (a, b)

Similarly we can check that with the remaining pairs (5,2), (5,3), (5,4), (5,6), (5,7) m is irrational, which is illustrated in the following table:

a	b	m
5	2	$\frac{10\pm\sqrt{76}}{6}$
5	3	$\frac{10\pm\sqrt{76}}{4}$
5	4	$\frac{10\pm\sqrt{84}}{2}$
5	6	$\frac{-10\pm\sqrt{124}}{2}$
5	7	$\frac{-10\pm\sqrt{156}}{4}$

Thus, we have show that when $a = 5$, b cannot be an integer less than 8.

Minimizing b

Given any value of a, what is the distinct minimum value of b that ensures a rational solution to the Box Problem?

a	b	r
5	8	**9**
6	16	50/3
7	15	**16**
8	15	49/3
9	24	**25**
10	16	18
11	35	**36**
12	32	100/3
13	48	**49**
15	24	27
17	80	**81**
19	99	**100**

MINIMIZING b

a	b	r
5	8	$9 = 3^2$
7	15	$16 = 4^2$
11	35	$36 = 6^2$
13	48	$49 = 7^2$
17	80	$81 = 9^2$
19	99	$100 = 10^2$
23	143	$169 = 13^2$
29	224	$289 = 17^2$

MARISSA'S CONJECTURE

When a is an odd prime of the form $2k + 1$, the minimum distinct value for b is $k^2 + 2k$. Also, when $a > 9$, and odd but not prime, $b = k^2 + 2k$ is the third minimum distinct value of b.

Bibliography

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