Difference Equations

Dan Hamilton
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**Difference Equations**

[#1] \( y_t = a_1 y_{t-1} \)  \hspace{1cm} First Order - homogenous

[#2] \( y_t = a_0 + a_1 y_{t-1} \)  \hspace{1cm} First Order (general)

[#3] \( y_t = a_1 y_{t-1} + a_2 y_{t-2} \)  \hspace{1cm} 2\textsuperscript{nd} Order – homogenous

[#4] \( y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} \)  \hspace{1cm} 2\textsuperscript{nd} Order (General)

A common rewriting: [#3'] \( y_t - a_1 y_{t-1} - a_2 y_{t-2} = 0 \)

**An example from Economics**

\[
i_t = \Delta k_{t+1} - \delta k_t
\]

**Net investment** \hspace{1cm} **Gross investment (capital purchases)**

**Losses due to depreciation**

Note: \( a_0 \), \( a_1 \), \( a_2 \) can be indexed by time “t”

**Lets rewrite this:**

\[
k_{t+1} - k_t - \delta k_t = i_t
\]

\[
k_{t+1} - (1 + \delta)k_t = i_t \hspace{1cm} \text{First Order (general)}
\]
Properties

The sizes of the coefficients on the lagged terms are very important.

Example: use [#1]

Let $y_0 = 1$ and $a_1 = 0.9$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$a_1 y_{t-1}$</th>
<th>$y_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-$</td>
<td>1.0</td>
</tr>
<tr>
<td>1</td>
<td>0.9(1)</td>
<td>0.9</td>
</tr>
<tr>
<td>2</td>
<td>0.9(0.9)</td>
<td>0.81</td>
</tr>
<tr>
<td>3</td>
<td>0.9(0.81)</td>
<td>0.72</td>
</tr>
<tr>
<td>4</td>
<td>0.9(0.72)</td>
<td>0.63</td>
</tr>
<tr>
<td>$\vdots$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>$\sim 0.03$</td>
<td></td>
</tr>
</tbody>
</table>

Let $y_0 = 1$ and $a_1 = 1.2$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$a_1 y_{t-1}$</th>
<th>$y_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-$</td>
<td>1.0</td>
</tr>
<tr>
<td>1</td>
<td>1.2(1)</td>
<td>1.2</td>
</tr>
<tr>
<td>2</td>
<td>1.2(1.2)</td>
<td>1.44</td>
</tr>
<tr>
<td>3</td>
<td>1.2(1.44)</td>
<td>1.7</td>
</tr>
<tr>
<td>4</td>
<td>1.2(1.7)</td>
<td>2.08</td>
</tr>
<tr>
<td>$\vdots$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td></td>
<td>200</td>
</tr>
</tbody>
</table>
Difference Equations

More Charts

$y_t = 0.9y_{t-1}$  
$y_t = 0.5y_{t-1}$

These are dynamically stable

$y_t = -0.5y_{t-1}$

It turns out that this one is unstable too

$y_t = y_{t-1} \ (a_1 = 1)$

These are “Dynamically Unstable”

$y_t = 1.2y_{t-1}$  
$y_t = -1.2y_{t-1}$

What is the punchline?
Stability Conditions

First Order: \[ |a_1| < 1 \]
Second Order: \[ a_1 + a_2 < 1 \] \[ a_2 < 1 + a_1 \] \[ |a_1| < 2 \] \[ -a_2 < 1 \text{ (if } a_2 \text{ is negative)} \]