Unconstrained Optimization:

1) Function of one variable: \( y = f(x) \)
   a) Maximization
      i. \( f'(x) = 0 \)
      ii. \( f''(x) \leq 0 \)
   b) Minimization
      i. \( f'(x) = 0 \)
      ii. \( f''(x) \geq 0 \)

2) Function of two variables: \( z = f(x, y) \)
   a) Maximization
      i. \( f_x = 0 \) and \( f_y = 0 \)
      ii. \( f_{xx} < 0, \ f_{yy} < 0 \) and \( f_{xx}f_{yy} - f_{xy}^2 > 0 \)
      iii. The second order condition implies that the second order total derivative of \( z, d^2z, \) is everywhere negative definite which further implies that the function \( z \) is strictly concave and hence a unique absolute maximum will exist.
      iv. If the strict inequality signs in the second order condition are changed to less than or equal to signs, \( \leq \), \( d^2z \) will be everywhere negative semidefinite which means that the function \( z \) is concave and hence multiple absolute maxima may exist.
   b) Minimization
      i. \( f_x = 0 \) and \( f_y = 0 \)
      ii. \( f_{xx} > 0, \ f_{yy} > 0 \) and \( f_{xx}f_{yy} - f_{xy}^2 > 0 \)
      iii. The second order condition implies that the second order total derivative of \( z, d^2z, \) is everywhere positive definite which further implies that the function \( z \) is strictly convex and hence a unique absolute minimum will exist.
iv. If the strict inequality signs in the second order condition are changed to greater than or equal to signs, \( \geq \), \( d^2z \) will be everywhere positive semidefinite which means that the function \( z \) is convex and hence **multiple** absolute minima may exist.

**Note:** For functions with two variables, if \( f_{xx}f_{yy} - f_{xy}^2 < 0 \) there exists a saddle point and thus no maximum or minimum exists.

3) Function of \( n \)-variables: \( z = f(x_1, x_2, \ldots, x_n) \)

a) Maximization

i. \( f_1 = f_2 = \cdots = f_n = 0 \)

ii. \( |H_1| < 0; |H_2| > 0; |H_3| < 0; \cdots; (-1)^n|H_n| > 0 \)

iii. Note: for maximization the principal minors of the Hessian alternate in sign with the first one being negative.

b) Minimization

i. \( f_1 = f_2 = \cdots = f_n = 0 \)

ii. \( |H_1| > 0; |H_2| > 0; |H_3| > 0; \cdots; |H_n| > 0 \)

iii. Note: for minimization the principal minors of the Hessian are all positive.

c) Note on Hessian Determinant

i. 
\[
|H| = \begin{vmatrix}
    f_{11} & f_{12} & f_{13} & \cdots & f_{1n} \\
    f_{21} & f_{22} & f_{23} & \cdots & f_{2n} \\
    f_{31} & f_{32} & f_{33} & \cdots & f_{3n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    f_{n1} & f_{n2} & f_{n3} & \cdots & f_{nn}
\end{vmatrix}
\]

ii. \( |H_1| = f_{11} \)

iii. \( |H_2| = \begin{vmatrix}
    f_{11} & f_{12} \\
    f_{21} & f_{22}
\end{vmatrix} \)

iv. \( |H_3| = \begin{vmatrix}
    f_{11} & f_{12} & f_{13} \\
    f_{21} & f_{22} & f_{23} \\
    f_{31} & f_{32} & f_{33}
\end{vmatrix} \)
v. To determine each successive principal minor of the Hessian Determinant add one column and one row to the previous principal minor.

Constrained Optimization with Equality Constraints:

4) One constraint:
   a) Set up
      i. Objective Function: \( z = f(x_1, x_2, \ldots, x_n) \)
      ii. Constraint: \( g(x_1, x_2, \ldots, x_n) = c \), where \( c \) is a constant
      iii. Lagrangian: \( Z = f(x_1, x_2, \ldots, x_n) + \lambda[c - g(x_1, x_2, \ldots, x_n)] \)
   b) Maximization
      i. \( Z_{\lambda} = Z_1 = Z_2 = \ldots = Z_n = 0 \)
      ii. \( |H_2| > 0; |H_3| < 0; |H_4| > 0; \ldots; (-1)^n |H_n| > 0 \)
   c) Minimization
      i. \( Z_{\lambda} = Z_1 = Z_2 = \ldots = Z_n = 0 \)
      ii. \( |H_2| < 0; |H_3| < 0; |H_4| < 0; \ldots; |H_n| < 0 \)
   d) Note on Bordered Hessian Determinant
      i. \( |H| = \begin{vmatrix} 0 & g_1 & g_2 & g_n \\ g_1 & Z_{11} & Z_{12} & Z_{1n} \\ g_2 & Z_{21} & Z_{22} & Z_{2n} \\ g_n & Z_{n1} & Z_{n2} & Z_{nn} \end{vmatrix} \)
      ii. \( |H_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & Z_{11} & Z_{12} \\ g_2 & Z_{21} & Z_{22} \end{vmatrix} \)
iii.  \[ |H_3| = \begin{vmatrix}
0 & g_1 & g_2 & g_3 \\
g_1 & Z_{11} & Z_{12} & Z_{13} \\
g_2 & Z_{21} & Z_{22} & Z_{23} \\
g_3 & Z_{31} & Z_{32} & Z_{33} 
\end{vmatrix} \]

iv.  To determine each successive principal minor of the Bordered Hessian Determinant add one column and one row to the previous principal minor.